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Equations of motion of a ground moving target for a multi-channel spaceborne SAR

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Defence R&D Canada – Ottawa

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Abstract

The modeling of a moving target for a single channel spaceborne SAR geometry has already been accomplished to a high degree of accuracy by Eldhuset *et al.*, but extending the model to include a SAR system that is equipped with multiple apertures (e.g. RADARSAT-2, TerraSAR-X) still requires further work. The purpose of this short technical memo is to do exactly that – to derive a set of equations of motion of a ground moving target for a multi-channel spaceborne SAR. These equations of motion will be shown to be applicable to both airborne and spaceborne multi-channel SAR systems in stripmap mode.

Résumé

Eldhuset et coll. ont déjà modélisé avec grande exactitude la géométrie de l’observation d’une cible mobile par un RSO monocanal spatioporté. Toutefois, de nouveaux travaux sont nécessaires pour adapter leur modèle à un RSO à ouvertures multiples (p. ex., RADARSAT-2 ou TerraSAR-X). Les travaux menant au présent mémoire technique visaient à dériver un ensemble d’équations de mouvement d’une cible terrestre mobile suivie par un RSO multicanal spatioporté. Nous montrons que l’on peut appliquer ces équations à l’observation en bande, autant par un RSO multicanal aéroporté que spatioporté.

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Executive summary

Equations of motion of a ground moving target for a multi-channel spaceborne SAR

S. Chiu, M.V. Dragošević; DRDC Ottawa TM 2008-326; Defence R&D Canada – Ottawa; March 2009.

Background: Both high resolution Synthetic Aperture Radar (SAR) processing and SAR Ground Moving Target Indication (SAR-GMTI) require that a highly accurate imaging geometry model be first established. This can be quite easily accomplished for the case of an airborne platform, which is assumed to be moving along a straight line and transmitting uniformly spaced pulses. This assumption requires good motion compensation and good control of the pulse repetition frequency (PRF) as a function of ground speed. For an orbiting platform, the gravitational force plays a key role in defining the satellite trajectory and, thus, must be taken into account. Also, equations of motion of a moving target that accurately model a SAR system equipped with multiple apertures (e.g. RADARSAT-2, TerraSAR-X) are evidently absent in the open literature, partly because there were no multi-aperture spaceborne SARs in the unclassified world prior to late 2007.

Principal results: The purpose of this short technical memo is to derive a set of equations of motion that accurately describes a ground moving target in spaceborne multi-channel SAR imaging geometry. A suitable multi-channel model of the target range history is derived in the Earth Centered, Earth-Fixed (ECEF) frame of reference using linearization for small angles as a function of the receive phase center index p (= 1, 2, 3, etc.) along the antenna and the pulse (or slow) time t:

$$
\begin{aligned}
R_p(t) = {} & R_0 + (p-1)D\,(\Psi - \Phi_0) + (V_{tr} - V_{sr})\,(t-t_0) \\
& + \frac{(p-1)D}{R_0}\,(V_{ta} - V_g)\,(t-t_0) + \frac{1}{2}\left(\frac{V_e^2 - 2V_sV_{ta}}{R_0} + A_{tr}\right)(t-t_0)^2
\end{aligned}
$$

The above range equation includes the following system parameters: R_0 is the target range at arbitrary time t_0; D is the distance between the effective phase centers; Ψ is the so-called squint angle between the radar broadside in the slant-range direction and the antenna pointing vector; Φ_0 is the direction of arrival angle (measured off-broadside from the SAR); V_{tr} and V_{sr} are the radial velocity components of the target and the SAR, respectively; V_{ta} is the target velocity component in the along-track or flight direction; V_s and V_g are the SAR orbital and ground speeds, respectively; V_e is the so-called effective velocity of the satellite; and A_{tr} is the target radial acceleration component. R_0, Φ_0, V_{tr}, V_{ta}, and A_{tr} are defined for time t_0. V_{sr} depends on V_s and Φ_0 in a predictable way. V_s, V_g, and V_e vary slowly with time and, therefore, may be evaluated anywhere in the neighborhood of t_0.

Significance of results: The accuracy of the equations of motion derived in this short memo has been tested and validated using the recently acquired RADARSAT-2 MODEX data. These equations of motion have been shown to be applicable to both airborne and spaceborne stripmap imaging geometries and serve as a physical basis for further algorithm development.

Sommaire

Equations of motion of a ground moving target for a multi-channel spaceborne SAR

S. Chiu, M.V. Dragošević ; DRDC Ottawa TM 2008-326 ; R & D pour la défense Canada – Ottawa ; mars 2009.

Introduction : Le traitement des données des radars à synthèse d'ouverture (RSO) de haute résolution et l'indication de cibles terrestres mobiles au sol nécessitent l'établissement préalable d'un modèle très précis de la géométrie de la visée. Ce modèle est facilement réalisé pour un RSO aéroporté pour lequel on peut postuler une trajectoire rectiligne et l'émission régulière d'impulsions. Pour être valides, ces deux hypothèses nécessitent une bonne compensation de mouvement et une régulation adéquate de la fréquence d'émission des impulsions en fonction de la vitesse au sol. Dans le cas d'une plateforme orbitale, la gravitation est un élément essentiel de la définition de la trajectoire et nous devons en tenir compte. Or, dans les sources publiées, on ne retrouve pas d'équations de mouvement d'une cible mobile permettant de modéliser avec exactitude un système RSO à ouvertures multiples (p. ex., RADARSAT-2 ou TerraSAR-X), notamment parce qu'avant la fin de 2007, il n'existait aucun système satellisé semblable dans le domaine non classifié.

Résultats : Le présent mémoire technique présente notre dérivation d'un ensemble d'équations de mouvement décrivant avec exactitude une cible terrestre mobile observée par un RSO multicanal spatioporté. Pour un repère géocentrique à axes fixes, nous avons dérivé un modèle adéquat de l'évolution des distances entre la cible et un RSO multicanal, par linéarisation des petits angles, en fonction de l'indice p (= 1, 2, 3, etc.) du centre de phase de réception le long de l'antenne et le temps t de l'impulsion (lent) :

$$\begin{aligned} R_p(t) = R_0 &+ (p-1)D\,(\Psi - \Phi_0) + (V_{tr} - V_{sr})\,(t - t_0) \\ &+ \frac{(p-1)D}{R_0}\,(V_{ta} - V_g)\,(t - t_0) + \frac{1}{2}\left(\frac{V_e^2 - 2V_sV_{ta}}{R_0} + A_{tr}\right)(t - t_0)^2 \end{aligned}$$

Les équations ci-dessus pour la distance comportent les paramètres suivants : R_0 est la distance de la cible au temps arbitraire t_0. D est la distance entre les centres effectifs de phase. Ψ correspond à l'angle de concentration entre la position de rayonnement transverse du radar dans le plan inclinaison-temps et le vecteur de pointage de l'antenne. Φ_0 représente la direction de l'angle d'incidence de l'écho (mesuré depuis le côté du RSO). V_{tr} et V_{sr} sont respectivement les vitesses radiales de la cible et du RSO. V_{ta} est la composante de la vitesse de la cible le long de la bande observée ou de la trajectoire de vol. V_s et V_g sont respectivement les vitesses du RSO en orbite et au sol. V_e est la vitesse effective du satellite. A_{tr} est la composante radiale de l'accélération de la cible. Les valeurs R_0, Φ_0, V_{tr}, V_{ta}, et

A_{tr} sont définies pour le temps t_0. V_{sr} dépend de V_s et Φ_0 de façon prévisible. Puisque V_s, V_g, et V_e varient lentement en fonction du temps, on peut les évaluer aux environs de t_0.

Portée : Nous avons testé et validé l'exactitude des équations de mouvement qui figurent dans le présent mémoire à l'aide de données RADARSAT-2 récemment acquises lors des essais MODEX. Nous avons démontré qu'elles peuvent s'appliquer à la géométrie de l'imagerie en bande (mode stripmap) par un radar aéroporté ou spatioporté. Elles constituent également un fondement tangible pour la mise au point d'autres algorithmes.

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1 Introduction

High resolution Synthetic Aperture Radar (SAR) processing requires that a highly accurate imaging geometry model be first established. For SAR Ground Moving Target Indication (SAR-GMTI), the underlying assumption that the radar scene is stationary must be extended to include non-stationary scenes or moving targets. This can be quite easily accomplished for the case of an airborne platform [1], which is assumed to be moving along a straight line and transmitting uniformly spaced pulses. This assumption requires good motion compensation and good control of the pulse repetition frequency (PRF) as a function of ground speed. The same cannot be said about a spaceborne platform, where the earth's gravitational force plays a key role in defining the platform trajectory and the velocity of the radar antenna footprint as it sweeps along the surface of the earth. The modeling of a moving target for a single channel spaceborne SAR geometry has already been accomplished to a high degree of accuracy by Eldhuset *et al.* [2] [3]. However, the extension of the model to include a SAR system that is equipped with multiple apertures is evidently absent in the open literature, partly because there were no existing spaceborne SAR systems in the unclassified world equipped with such a capability until the launches of TerraSAR-X and RADARSAT-2 in late 2007. The purpose of this short technical memo is to derive a set of equations of motion of a ground moving target for a multi-channel spaceborne SAR. These equations of motion will be shown to be applicable to both airborne and spaceborne stripmap imaging geometries and serve as a physical basis for further algorithm development.

2 Equations of Motion of a Moving Target

The relative position vector of a moving target with respect to an imaging SAR satellite, in the earth centered earth fixed (ECEF) system, can be written as

$$\mathbf{R} = \mathbf{R}_t - \mathbf{R}_s \,, \tag{1}$$

where indices t and s denote 'target' and 'satellite,' respectively, and bold letters indicate vectors. The magnitude of a vector is represented by the same symbol with the regular italic font. In the ECEF frame, the earth motion is absorbed into the relative satellite motion.

The Doppler centroid and Doppler rate are proportional to $\dot{R}$ and $\ddot{R}$, respectively, where the dot and double-dot notations indicate first and second derivatives with respect to time. A common approach to the derivation of $\dot{R}$ and $\ddot{R}$ is to start from the identity [3]

$$R^2 = \mathbf{R}^T \mathbf{R} \,, \tag{2}$$

and to differentiate it with respect to time. $(\bullet)^T$ denotes the vector (or matrix) transpose.

Differentiating both sides of (2) with respect to time, we get

$$
\begin{aligned}
2R\dot{R} &= \dot{\mathbf{R}}^T\mathbf{R} + \mathbf{R}^T\dot{\mathbf{R}} &\quad (3a)\\
\dot{R} &= \frac{\dot{\mathbf{R}}^T\mathbf{R}}{R} = \frac{\mathbf{R}^T\dot{\mathbf{R}}}{R}\,, &\quad (3b)
\end{aligned}
$$

where $\mathbf{A}^T\mathbf{B} = \mathbf{B}^T\mathbf{A}$ is obviously true for any two vectors $\mathbf{A}$ and $\mathbf{B}$.

Equation (3b) can be rewritten as

$$
\begin{aligned}
\dot{R} &= \frac{\mathbf{R}^T}{R}(\dot{\mathbf{R}}_t - \dot{\mathbf{R}}_s) &\quad (4a)\\
&= \frac{\mathbf{R}^T}{R}(\mathbf{V}_t - \mathbf{V}_s) &\quad (4b)\\
&= V_{tr} - V_{sr}, &\quad (4c)
\end{aligned}
$$

where $\mathbf{V}_t \equiv \dot{\mathbf{R}}_t$ is the velocity vector of the moving target, $\mathbf{V}_s \equiv \dot{\mathbf{R}}_s$ is the velocity vector of the satellite, and

$$
\begin{aligned}
V_{tr} &\equiv \left(\frac{\mathbf{R}}{R}\right)^T \mathbf{V}_t &\quad (5a)\\
V_{sr} &\equiv \left(\frac{\mathbf{R}}{R}\right)^T \mathbf{V}_s\,, &\quad (5b)
\end{aligned}
$$

are the projections of the target and satellite velocity vectors onto the line of sight (LOS) or the radial direction, respectively. Also, the radial speed of a stationary target as "seen" by the radar due to the platform motion is equal to $-V_{sr}$. Therefore, the Doppler shift at the beam center induced by the motion of the platform (or the stationary clutter Doppler centroid) is given by

$$f_{DC} = 2\frac{V_{sr}}{\lambda}\,, \qquad (6)$$

and the Doppler shift due to the target's radial speed is

$$f_{dc} = -2\frac{V_{tr}}{\lambda}\,. \qquad (7)$$

Therefore, the total Doppler shift is given by

$$F_{DC} = -2\frac{V_r}{\lambda} = -2\frac{(V_{tr} - V_{sr})}{\lambda}\,. \qquad (8)$$

Again, differentiating both sides of (3a) with respect to time yields

$$
\begin{aligned}
2(\dot{R}^2 + R\ddot{R}) &= \ddot{\mathbf{R}}^T\mathbf{R} + \dot{\mathbf{R}}^T\dot{\mathbf{R}} + \dot{\mathbf{R}}^T\dot{\mathbf{R}} + \mathbf{R}^T\ddot{\mathbf{R}} &\quad (9a)\\
&= 2\mathbf{R}^T\ddot{\mathbf{R}} + 2\dot{\mathbf{R}}^T\dot{\mathbf{R}} &\quad (9b)\\
R\ddot{R} &= \mathbf{R}^T\ddot{\mathbf{R}} + \dot{\mathbf{R}}^T\dot{\mathbf{R}} - \dot{R}^2\,. &\quad (9c)
\end{aligned}
$$

Using the following definitions

$$\begin{aligned}
\mathbf{V}_t &\equiv \dot{\mathbf{R}}_t && \text{(10a)}\\
\mathbf{V}_s &\equiv \dot{\mathbf{R}}_s && \text{(10b)}\\
\mathbf{A}_t &\equiv \ddot{\mathbf{R}}_t && \text{(10c)}\\
\mathbf{A}_s &\equiv \ddot{\mathbf{R}}_s && \text{(10d)}\\
\mathbf{A} &\equiv \ddot{\mathbf{R}} = \ddot{\mathbf{R}}_t - \ddot{\mathbf{R}}_s = \mathbf{A}_t - \mathbf{A}_s && \text{(10e)}\\
A_{tr} &\equiv \mathbf{R}^T \mathbf{A}_t / R \,, && \text{(10f)}
\end{aligned}$$

(9c) can be rewritten as

$$\begin{aligned}
R\ddot{R} &= R\left(\frac{\mathbf{R}^T}{R}\mathbf{A}_t\right) - \mathbf{R}^T\mathbf{A}_s + (\mathbf{V}_t - \mathbf{V}_s)^T(\mathbf{V}_t - \mathbf{V}_s) - (V_{tr} - V_{sr})^2 && \text{(11a)}\\
&= RA_{tr} - \mathbf{R}^T\mathbf{A}_s + V_t^2 - \mathbf{V}_t^T\mathbf{V}_s - \mathbf{V}_s^T\mathbf{V}_t + V_s^2 - (V_{tr}^2 - 2V_{tr}V_{sr} + V_{sr}^2) && \text{(11b)}\\
&= V_s^2 - \mathbf{R}^T\mathbf{A}_s + RA_{tr} + V_t^2 - 2V_s\left(\frac{\mathbf{V}_s^T}{V_s}\mathbf{V}_t\right) - V_{tr}^2 + 2V_{tr}V_{sr} - V_{sr}^2 \,. && \text{(11c)}
\end{aligned}$$

Therefore,

$$\ddot{R} = \frac{V_e^2}{R} - \frac{V_{sr}^2}{R} + A_{tr} + \frac{V_t^2}{R} - \frac{2V_sV_{ta}}{R} - \frac{V_{tr}^2}{R} + \frac{2V_{tr}V_{sr}}{R} \,, \tag{12}$$

where

$$\begin{aligned}
V_e^2 &\equiv V_s^2 - \mathbf{R}^T\mathbf{A}_s && \text{(13a)}\\
V_{ta} &= \left(\frac{\mathbf{V}_s}{V_s}\right)^T \mathbf{V}_t \,. && \text{(13b)}
\end{aligned}$$

V_e is the so-called "effective velocity" often used in the spaceborne SAR processing to model the range equation and V_{ta} is the projection of the target velocity onto the direction of platform velocity $\mathbf{V}_s$, which is also called the along-track direction and may not necessarily be parallel to the ground track.

The instantaneous slant range equation (or history) $R(t)$ is the key to high precision SAR processing. The accurate estimation of the effective velocity in the equation allows complicated mathematical manipulations involving a satellite/earth geometry model to be avoided and a simple hyperbolic approximation to be adopted in most high precision SAR processing algorithms. The hyperbolic model can be further simplified and approximated using a second order Taylor series expansion or a parabolic model without significantly incurring further loss of accuracy, considering typical RADARSAT-2 dwell times and resolution. This may not be true in general.

If $\dot{R}$ and $\ddot{R}$ in (4a) and (12) are evaluated at some arbitrary time t_0, then the range equation can be approximated by the Taylor series expansion:

$$R(t) \approx R_0 + V_{r0}(t - t_0) + \frac{A_{r0}}{2}(t - t_0)^2 \,, \tag{14}$$

where

$$\begin{aligned}
R_0 &= R(t_0) = |\mathbf{R}_t(t_0) - \mathbf{R}_s(t_0)| && \text{(15a)}\\
V_{r0} &= \dot{R}(t_0) = \frac{\mathbf{R}_0^T}{R_0}[\mathbf{V}_t(t_0) - \mathbf{V}_s(t_0)] = V_{tr} - V_{sr} && \text{(15b)}\\
A_{r0} &= \ddot{R}(t_0) = \frac{V_e^2 - V_{sr}^2}{R_0} + \frac{V_t^2 - V_{tr}^2 + 2V_{tr}V_{sr} - 2V_sV_{ta}}{R_0} + A_{tr} && \text{(15c)}\\
&\approx \frac{V_e^2 - V_{sr}^2}{R_0} + \frac{2V_{tr}V_{sr} - 2V_sV_{ta}}{R_0} + A_{tr}\ . && \text{(15d)}
\end{aligned}$$

V_t^2 and V_{tr}^2 are considered negligible with respect to '$2V_sV_{ta}$' and, therefore, are dropped in (15d). If t_0 is chosen to be the broadside time t_b, then $V_{sr} = 0$ by definition and the radial direction (subscripted by 'r') becomes exactly perpendicular to the flight direction or the along-track direction (subscripted by 'a'). Under this condition, (15b) and (15d) become

$$\begin{aligned}
V_{rb} &= \dot{R}(t_b) = \frac{\mathbf{R}_b^T}{R_b}\mathbf{V}_t(t_b) = V_{tr} && \text{(16a)}\\
A_{rb} &\approx \ddot{R}(t_b) = \frac{V_e^2 - 2V_sV_{ta}}{R_b} + A_{tr}\ , && \text{(16b)}
\end{aligned}$$

where V_{tr} and A_{tr} are now the target's down-range (or across-track) velocity and acceleration components, respectively, and R_b is the broadside range of the moving target.

3 Two-Channel SAR

The use of a parabolic model is useful in the derivation of range equations for multi-channel SAR systems. In this section the range equation for the second aperture of a two-channel SAR is derived.

3.1 Local Frame of Reference

In order to continue with our derivations, we first define a local flight (LF) frame of reference for the radar as shown in Figure 1, where $\mathbf{d}$ is defined as the unit vector pointing down from the radar's center of gravity to the center of the earth. To define the second axis, we vector cross-multiply $\mathbf{d}$ with the radar's velocity vector $\mathbf{V}_s$ to form the right pointing unit vector $\mathbf{r}$:

$$\mathbf{r} = \frac{\mathbf{d}\times\mathbf{V}_s}{|\mathbf{d}\times\mathbf{V}_s|}\ . \tag{17}$$

Then the third unit vector, which completes the local reference frame, is given by

$$\mathbf{f} = \mathbf{r}\times\mathbf{d}\ . \tag{18}$$

We should point out that $\mathbf{V}_s$ is not necessarily in the exact same direction as $\mathbf{f}$, as illustrated in Figure 1

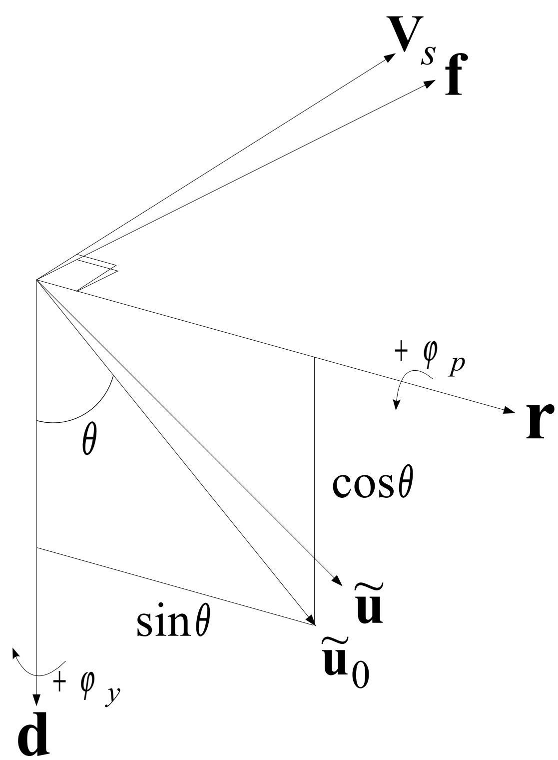

Figure 1: *Local reference frame of radar.*

3.2 Transformation Matrix

We now derive the transformation matrix from the LF reference frame to the ECEF reference frame. To begin, we express the unit vector $\mathbf{d}$ in the ECEF frame:

$$\mathbf{d} = -\frac{\mathbf{R}_s}{|\mathbf{R}_s|} \tag{19a}$$

$$= -\frac{1}{R_s}\begin{bmatrix} R_{sx} \\ R_{sy} \\ R_{sz} \end{bmatrix}. \tag{19b}$$

Then $\mathbf{r}$ becomes

$$\mathbf{r} = \frac{\mathbf{d}\times\mathbf{V}_s}{|\mathbf{d}\times\mathbf{V}_s|} \tag{20a}$$

$$= \frac{\mathbf{d}\times\mathbf{V}_s}{V_{hor}} \tag{20b}$$

$$= \frac{1}{R_s V_{hor}}\begin{bmatrix} R_{sz}V_{sy}-R_{sy}V_{sz} \\ R_{sx}V_{sz}-R_{sz}V_{sx} \\ R_{sy}V_{sx}-R_{sx}V_{sy} \end{bmatrix}, \tag{20c}$$

where

$$V_{hor} = |\mathbf{d}\times\mathbf{V}_s| \tag{21a}$$

$$\mathbf{V}_s = \begin{bmatrix} V_{sx} \\ V_{sy} \\ V_{sz} \end{bmatrix}. \tag{21b}$$

V_{hor} is obviously the horizontal velocity component of the radar and can be easily shown to be

$$V_{hor} = \sqrt{V_{sx}^2+V_{sy}^2+V_{sz}^2-V_{ver}^2}\,, \tag{22}$$

where V_{ver} is the vertical velocity component of the radar platform and is given by

$$V_{ver} = \frac{\mathbf{R}_s^T}{R_s}\mathbf{V}_s = \frac{R_{sx}V_{sx}+R_{sy}V_{sy}+R_{sz}V_{sz}}{R_s}\,. \tag{23}$$

We are now ready to express the forward unit vector $\mathbf{f}$ in the ECEF frame as

$$\mathbf{f} = \mathbf{r}\times\mathbf{d} = -\mathbf{d}\times\mathbf{r} \tag{24a}$$

$$= \frac{1}{R_s}\begin{bmatrix} R_{sx} \\ R_{sy} \\ R_{sz} \end{bmatrix} \times \frac{1}{R_s V_{hor}}\begin{bmatrix} R_{sz}V_{sy}-R_{sy}V_{sz} \\ R_{sx}V_{sz}-R_{sz}V_{sx} \\ R_{sy}V_{sx}-R_{sx}V_{sy} \end{bmatrix} \tag{24b}$$

$$= \frac{1}{R_s V_{hor}}\begin{bmatrix} R_sV_{sx}-R_{sx}V_{ver} \\ R_sV_{sy}-R_{sy}V_{ver} \\ R_sV_{sz}-R_{sz}V_{ver} \end{bmatrix}. \tag{24c}$$

Finally, the transformation matrix from the LF reference frame to the ECEF frame [4] is simply

$$\begin{aligned}
\mathbf{\Gamma}_f &= \begin{bmatrix} \mathbf{f} & \mathbf{r} & \mathbf{d} \end{bmatrix} && \text{(25a)}\\
&= \frac{1}{R_s V_{hor}} \begin{bmatrix} R_s V_{sx} - R_{sx} V_{ver} & R_{sz} V_{sy} - R_{sy} V_{sz} & -R_{sx} V_{hor} \\ R_s V_{sy} - R_{sy} V_{ver} & R_{sx} V_{sz} - R_{sz} V_{sx} & -R_{sy} V_{hor} \\ R_s V_{sz} - R_{sz} V_{ver} & R_{sy} V_{sx} - R_{sx} V_{sy} & -R_{sz} V_{hor} \end{bmatrix}. && \text{(25b)}
\end{aligned}$$

3.3 Antenna Look Vector

Let the ideal look direction of the antenna in the LF frame, with an off-nadir angle θ pointing at a zero Doppler point on the surface of the earth, be

$$\tilde{\mathbf{u}}_0 = \begin{bmatrix} 0 \\ \sin\theta \\ \cos\theta \end{bmatrix}. \tag{26}$$

Then the actual antenna look vector (or pointing vector) in the local reference frame of the radar is given by

$$\begin{aligned}
\tilde{\mathbf{u}} &= \mathbf{\Gamma}_\varphi \tilde{\mathbf{u}}_0 && \text{(27a)}\\
&\approx \begin{bmatrix} -\varphi_y \sin\theta + \varphi_p \cos\theta \\ \sin\theta \\ \cos\theta \end{bmatrix}, && \text{(27b)}
\end{aligned}$$

where $\mathbf{\Gamma}_\varphi$ is the yaw-pitch rotation matrix, and φ_y and φ_p are the yaw and pitch angles about the axes $\mathbf{d}$ and $\mathbf{r}$, respectively. We are assuming that φ_y and φ_p correspond to a LOS within the beam, but not necessarily at its center. For RADARSAT-2, φ_y and φ_p are typically small ($\ll 1$) in the ECEF frame due to the zero-Doppler beam steering. The rotation matrix $\mathbf{\Gamma}_\varphi$ can, therefore, be shown as

$$\begin{aligned}
\mathbf{\Gamma}_\varphi = \mathbf{M}_r \mathbf{M}_d &= \begin{bmatrix} 1 & -\varphi_y & \varphi_p \\ \varphi_y & 1 & \varphi_y \varphi_p \\ -\varphi_p & 0 & 1 \end{bmatrix} && \text{(28a)}\\
&\approx \begin{bmatrix} 1 & -\varphi_y & \varphi_p \\ \varphi_y & 1 & 0 \\ -\varphi_p & 0 & 1 \end{bmatrix}, && \text{(28b)}
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{M}_r &= \begin{bmatrix} \cos\varphi_p & 0 & \sin\varphi_p \\ 0 & 1 & 0 \\ -\sin\varphi_p & 0 & \cos\varphi_p \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & \varphi_p \\ 0 & 1 & 0 \\ -\varphi_p & 0 & 1 \end{bmatrix} && \text{(29a)}\\
\mathbf{M}_d &= \begin{bmatrix} \cos\varphi_y & -\sin\varphi_y & 0 \\ \sin\varphi_y & \cos\varphi_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & -\varphi_y & 0 \\ \varphi_y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. && \text{(29b)}
\end{aligned}$$

The term, $\varphi_y\varphi_p$, is considered negligible [4] and is set to zero in (27b) and (28b). In the ECEF frame, the antenna look vector $\mathbf{u}$ is then given by

$$\mathbf{u} = \mathbf{\Gamma}_f \tilde{\mathbf{u}} = \mathbf{\Gamma}_f \mathbf{\Gamma}_\varphi \tilde{\mathbf{u}}_0 \; . \tag{30}$$

We should note that the look vector $\mathbf{u}$ is not necessarily in the direction of the beam center, rather it points to the direction of the target of interest within the beam footprint.

3.4 Displacement Vector

Let $\tilde{\mathbf{D}}$ denote the vector pointing from the effective phase center of the aft sub-aperture to the effective phase center of the fore sub-aperture in the LF frame, $\tilde{\mathbf{D}}$ can then be expressed as

$$\tilde{\mathbf{D}} = \mathbf{\Gamma}_\psi \begin{bmatrix} D \\ 0 \\ 0 \end{bmatrix} \approx D \begin{bmatrix} 1 \\ \psi_y \\ -\psi_p \end{bmatrix} , \tag{31}$$

where

$$\mathbf{\Gamma}_\psi = \mathbf{\Gamma}_\varphi(\varphi = \psi) \approx \begin{bmatrix} 1 & -\psi_y & \psi_p \\ \psi_y & 1 & 0 \\ -\psi_p & 0 & 1 \end{bmatrix} , \tag{32}$$

and ψ_y and ψ_p are the pitch and yaw angles (or the orientation) of the antenna, representing the attitude of the spacecraft in the LF frame of reference. In the ECEF frame, $\tilde{\mathbf{D}}$ becomes

$$\mathbf{D} = \mathbf{\Gamma}_f \tilde{\mathbf{D}} = D\mathbf{\Gamma}_f \begin{bmatrix} 1 \\ \psi_y \\ -\psi_p \end{bmatrix} . \tag{33}$$

3.5 Range Equations for Multiple Phase Centers

A two-aperture SAR-MTI system is again assumed in the following derivations with the understanding that the derived equations can be generalized to a multi-aperture system. Let $\mathbf{R}_{s1}$ and $\mathbf{R}_{s2}$ denote the position vectors of the antenna's two effective (or two-way) phase centers in the ECEF frame, respectively. The aft antenna phase center $\mathbf{R}_{s2}$ is then displaced from the fore antenna phase center $\mathbf{R}_{s1}$ by $-\mathbf{D}$. For the case of RADARSAT-2, the displacement vector $\mathbf{D}$ is closely aligned with the radar's velocity vector $\mathbf{V}_s$. Perfect alignment would be optimal because it would allow the aft phase center to pass through the same ECEF position as the fore phase center with a time delay of D/V_s, where D is the distance between the two effective phase centers. This perfect alignment would also mean that the whole antenna is ideally steered, generating a zero Doppler centroid in the clutter Doppler spectrum. In the presence of a non-zero Doppler centroid, there exists a non-zero across-track component of $\mathbf{D}$, which translates into a small across-track baseline.

In the case of a real spaceborne SAR-MTI system, such as the RADARSAT-2 MODEX, this small cross-track component is always present and, therefore, must be compensated for or taken into account in the system modeling.

The slant-range vector $\mathbf{R}_2$ from the aft antenna phase center to the target can, therefore, be expressed as

$$\begin{aligned} \mathbf{R}_2 &= \mathbf{R}_t - \mathbf{R}_{s2} && (34a)\\ &= \mathbf{R}_t - (\mathbf{R}_{s1} - \mathbf{D}) && (34b)\\ &= \mathbf{R}_1 + \mathbf{D}\,, && (34c) \end{aligned}$$

where $\mathbf{R}_1 = \mathbf{R}_t - \mathbf{R}_{s1}$ and $\mathbf{R}_{s2} = \mathbf{R}_{s1} - \mathbf{D}$. Then the projections of these slant-range vectors, $\mathbf{R}_1$ and $\mathbf{R}_2$, along the look vector $\mathbf{u}$ direction are given by

$$\begin{aligned} R_1 &= \mathbf{R}_1^T\mathbf{u} && (35a)\\ R_2 &= \mathbf{R}_2^T\mathbf{u} = (\mathbf{R}_1^T + \mathbf{D}^T)\mathbf{u} = R_1 + \mathbf{D}^T\mathbf{u} && (35b)\\ &= R_1 + (\mathbf{\Gamma}_f\tilde{\mathbf{D}})^T\mathbf{\Gamma}_f\tilde{\mathbf{u}} = R_1 + \tilde{\mathbf{D}}^T\mathbf{\Gamma}_f^T\mathbf{\Gamma}_f\tilde{\mathbf{u}} && (35c)\\ &= R_1 + \tilde{\mathbf{D}}^T\tilde{\mathbf{u}}\,. && (35d) \end{aligned}$$

From (27b) and (31), (35d) becomes

$$\begin{aligned} R_2(t) &\approx R_1(t) + D\begin{bmatrix} 1 & \psi_y & -\psi_p \end{bmatrix}\begin{bmatrix} -\varphi_y\sin\theta + \varphi_p\cos\theta \\ \sin\theta \\ \cos\theta \end{bmatrix} && (36a)\\ &= R_1(t) + D[(\psi_y - \varphi_y)\sin\theta - (\psi_p - \varphi_p)\cos\theta] && (36b)\\ &= R_1(t) + D(\Psi - \Phi)\,, && (36c) \end{aligned}$$

where

$$\begin{aligned} \Psi &= \psi_y\sin\theta - \psi_p\cos\theta && (37a)\\ \Phi &= \varphi_y\sin\theta - \varphi_p\cos\theta\,. && (37b) \end{aligned}$$

Ψ and Φ are now measured in the slant-range plane. As the antenna footprint sweeps across the target, the pitch angle φ_p hardly changes (i.e. remains virtually constant) such that $\varphi_p \approx \psi_p$, resulting in $\Psi - \Phi \approx (\psi_y - \varphi_y)\sin\theta$. In the case of RADARSAT-2, ψ_y and ψ_p are usually small but non-zero such that the beam center is not located exactly at the zero-Doppler point on the surface of the earth (in the ECEF frame). This residual beam squint Ψ generates a small constant along-track interferometric phase, which is usually removed by the digital-balance processing of the signal channels and can, therefore, be ignored. For the sake of completeness, however, we shall keep the term in (36c). Then, the zeroth-order coefficient of the Taylor expansion of $R_2(t)$ evaluated at arbitrary time t_0 can be expressed as

$$R_2(t_0) = R_1(t_0) - D\left[\Phi(t_0) - \Psi\right]\,. \tag{38}$$

Next, we derive the first-order coefficient of the Taylor series expansion of $R_2(t)$. From (34c), we obtain

$$
\begin{aligned}
R_2^2 &= (\mathbf{R}_1+\mathbf{D})^T(\mathbf{R}_1+\mathbf{D}) && \text{(39a)}\\
R_2\dot{R}_2 &= (\mathbf{R}_1+\mathbf{D})^T(\dot{\mathbf{R}}_1+\dot{\mathbf{D}}) && \text{(39b)}\\
\dot{R}_2 &= \frac{\mathbf{R}_1^T\dot{\mathbf{R}}_1+\mathbf{R}_1^T\dot{\mathbf{D}}+\mathbf{D}^T\dot{\mathbf{R}}_1+\mathbf{D}^T\dot{\mathbf{D}}}{R_2} && \text{(39c)}\\
&= \frac{R_1\dot{R}_1}{R_2}+\frac{\mathbf{R}_1^T\dot{\mathbf{D}}}{R_2}+\frac{\mathbf{D}^T\dot{\mathbf{R}}_1}{R_2}+\frac{\mathbf{D}^T\dot{\mathbf{\Gamma}}_f\tilde{\mathbf{D}}}{R_2} && \text{(39d)}\\
&\approx \dot{R}_1+\frac{\mathbf{R}^T}{R}\dot{\mathbf{D}}+\frac{\mathbf{D}^T}{R}\dot{\mathbf{R}}+\frac{O(D^2)}{R} && \text{(39e)}\\
&\approx \dot{R}_1+\mathbf{u}^T\dot{\mathbf{D}}+\frac{\mathbf{D}^T}{R}(\mathbf{V}_t-\mathbf{V}_s)\,, && \text{(39f)}
\end{aligned}
$$

where it can be shown that $\mathbf{R}_1^T\dot{\mathbf{R}}_1 = R_1\dot{R}_1$, $R_1 \approx R_2 = R$, and the $O(D^2)$ term can be neglected.

First, we derive the second term in (39f):

$$
\mathbf{u}^T\dot{\mathbf{D}} = \mathbf{u}^T\frac{\partial}{\partial t}\left(\mathbf{\Gamma}_f\tilde{\mathbf{D}}\right) = \mathbf{u}^T\left(\dot{\mathbf{\Gamma}}_f\tilde{\mathbf{D}}+\mathbf{\Gamma}_f\dot{\tilde{\mathbf{D}}}\right) = \mathbf{u}^T\dot{\mathbf{\Gamma}}_f\tilde{\mathbf{D}}\,, \tag{40}
$$

where we have assumed that the spacecraft attitude is not changing in the LF frame such that time derivatives of ψ_y and ψ_p (or $\dot{\tilde{\mathbf{D}}}$) are equal to zero in the imaging time interval. We also assume, for simplicity, that ψ_y and ψ_p are small. This is normally true for RADARSAT-2, which utilizes dynamic antenna squinting to constantly steer the beam to a zero Doppler point on the surface of the earth. Therefore, (40) becomes

$$
\mathbf{u}^T\dot{\mathbf{D}} = \mathbf{u}^T\dot{\mathbf{\Gamma}}_f D\begin{bmatrix}1\\ \psi_y\\ -\psi_p\end{bmatrix} \approx \mathbf{u}^T\dot{\mathbf{\Gamma}}_f D\begin{bmatrix}1\\0\\0\end{bmatrix}. \tag{41}
$$

Here, we need to find the first time derivative of $\mathbf{\Gamma}_f$ (or $\dot{\mathbf{\Gamma}}_f$), which can be shown to be

$$
\dot{\mathbf{\Gamma}}_f = \frac{1}{R_sV_{hor}}\begin{bmatrix}
\dot{R}_sV_{sx}+R_sA_{sx}-V_{sx}V_{ver}-R_{sx}\dot{V}_{ver} & R_{sz}A_{sy}-R_{sy}A_{sz} & -V_{sx}V_{hor}-R_{sx}\dot{V}_{hor}\\
\dot{R}_sV_{sy}+R_sA_{sy}-V_{sy}V_{ver}-R_{sy}\dot{V}_{ver} & R_{sx}A_{sz}-R_{sz}A_{sx} & -V_{sy}V_{hor}-R_{sy}\dot{V}_{hor}\\
\dot{R}_sV_{sz}+R_sA_{sz}-V_{sz}V_{ver}-R_{sz}\dot{V}_{ver} & R_{sy}A_{sx}-R_{sx}A_{sy} & -V_{sz}V_{hor}-R_{sz}\dot{V}_{hor}
\end{bmatrix}. \tag{42}
$$

where terms of the type $V_{sx}V_{sy}$, $V_{sx}V_{sz}$, and $V_{sz}V_{sy}$ cancel out in the second column of (42) and are, therefore, dropped. We can further simplify (42) by noting that $\dot{R}_s \approx 0$, $\dot{V}_{hor} \approx 0$, and $\dot{V}_{ver} \approx 0$:

$$
\dot{\mathbf{\Gamma}}_f \approx \frac{1}{R_sV_{hor}}\begin{bmatrix}
R_sA_{sx}-V_{sx}V_{ver} & R_{sz}A_{sy}-R_{sy}A_{sz} & -V_{sx}V_{hor}\\
R_sA_{sy}-V_{sy}V_{ver} & R_{sx}A_{sz}-R_{sz}A_{sx} & -V_{sy}V_{hor}\\
R_sA_{sz}-V_{sz}V_{ver} & R_{sy}A_{sx}-R_{sx}A_{sy} & -V_{sz}V_{hor}
\end{bmatrix}. \tag{43}
$$

Therefore, (41) becomes

$$\mathbf{u}^T \dot{\mathbf{D}} \approx \frac{D}{R_s V_{hor}} \mathbf{u}^T \begin{bmatrix} R_s A_{sx} - V_{sx} V_{ver} \\ R_s A_{sy} - V_{sy} V_{ver} \\ R_s A_{sz} - V_{sz} V_{ver} \end{bmatrix} \tag{44a}$$

$$= \frac{D}{V_{hor}} \mathbf{u}^T \mathbf{A}_s - \frac{D V_{ver}}{R_s V_{hor}} \mathbf{u}^T \mathbf{V}_s \tag{44b}$$

$$\approx \frac{D}{V_{hor}} \mathbf{u}^T \mathbf{A}_s \,. \tag{44c}$$

The last term in (44b) is very small, since the look vector $\mathbf{u}$ is virtually perpendicular to $\mathbf{V}_s$ (i.e. $\mathbf{u} \perp \mathbf{V}_s$) in the ECEF frame, and is, therefore, ignored.

We now derive the last term of (39f). From (25b) and (33),

$$\frac{\mathbf{D}^T}{R} (\mathbf{V}_t - \mathbf{V}_s)$$

$$= \frac{D}{R} \begin{bmatrix} 1 & \psi_y & -\psi_p \end{bmatrix} \boldsymbol{\Gamma}_f^T \left(\mathbf{V}_t - \begin{bmatrix} V_{sx} \\ V_{sy} \\ V_{sz} \end{bmatrix} \right) \tag{45a}$$

$$\approx \frac{D \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}{R R_s V_{hor}} \begin{bmatrix} R_s V_{sx} - R_{sx} V_{ver} & R_s V_{sy} - R_{sy} V_{ver} & R_s V_{sz} - R_{sz} V_{ver} \\ R_{sz} V_{sy} - R_{sy} V_{sz} & R_{sx} V_{sz} - R_{sz} V_{sx} & R_{sy} V_{sx} - R_{sx} V_{sy} \\ -R_{sx} V_{hor} & -R_{sy} V_{hor} & -R_{sz} V_{hor} \end{bmatrix} \times \left(\mathbf{V}_t - \begin{bmatrix} V_{sx} \\ V_{sy} \\ V_{sz} \end{bmatrix} \right) \tag{45b}$$

$$= \frac{D \begin{bmatrix} R_s V_{sx} - R_{sx} V_{ver} & R_s V_{sy} - R_{sy} V_{ver} & R_s V_{sz} - R_{sz} V_{ver} \end{bmatrix}}{R R_s V_{hor}} \left(\mathbf{V}_t - \begin{bmatrix} V_{sx} \\ V_{sy} \\ V_{sz} \end{bmatrix} \right), \tag{45c}$$

where $\psi_y \ll 1$, $\psi_p \ll 1$, and they are neglected in (45b). Also by noting that

$$R_s \mathbf{V}_s^T - V_{ver} \mathbf{R}_s^T = \begin{bmatrix} R_s V_{sx} - R_{sx} V_{ver} & R_s V_{sy} - R_{sy} V_{ver} & R_s V_{sz} - R_{sz} V_{ver} \end{bmatrix} \tag{46a}$$

$$\mathbf{R}_s = \mathbf{R}_s + \mathbf{R}_t - \mathbf{R}_t = \mathbf{R} + \mathbf{R}_t \tag{46b}$$

$$\mathbf{V}_s^T \mathbf{V}_t = V_s V_{ta} \tag{46c}$$

$$(\mathbf{R} + \mathbf{R}_t)^T \mathbf{V}_t = R V_{tr} + \mathbf{R}_t^T \mathbf{V}_t \approx R V_{tr} \,, \tag{46d}$$

we can rewrite (45c) as

$$\begin{aligned}\frac{\mathbf{D}^T}{R}(\mathbf{V}_t-\mathbf{V}_s)\approx&\frac{D}{RR_sV_{hor}}\left[\left(R_s\mathbf{V}_s^T-V_{ver}\mathbf{R}_s^T\right)\mathbf{V}_t\right]\\&+\frac{D}{RR_sV_{hor}}\left[-R_sV_s^2+V_{ver}\left(R_{sx}V_{sx}+R_{sy}V_{sy}+R_{sz}V_{sz}\right)\right] &(47a)\\=&\frac{D}{RR_sV_{hor}}\left[\left(R_s\mathbf{V}_s^T-V_{ver}\mathbf{R}_s^T\right)\mathbf{V}_t-R_sV_s^2+V_{ver}\mathbf{R}_s^T\mathbf{V}_s\right] &(47b)\\=&\frac{D}{RR_sV_{hor}}\left[R_s\mathbf{V}_s^T\mathbf{V}_t-V_{ver}\left(\mathbf{R}+\mathbf{R}_t\right)^T\mathbf{V}_t-R_sV_s^2+V_{ver}^2R_s\right] &(47c)\\=&\frac{D}{RR_sV_{hor}}\left(R_sV_sV_{ta}-RV_{ver}V_{tr}-R_sV_s^2+R_sV_{ver}^2\right) &(47d)\\=&D\left[\left(\frac{V_s}{V_{hor}}\right)\frac{V_{ta}}{R}-\left(\frac{V_{ver}}{V_{hor}}\right)\frac{V_{tr}}{R_s}-\left(\frac{V_s}{V_{hor}}\right)\frac{V_s}{R}+\left(\frac{V_{ver}}{V_{hor}}\right)\frac{V_{ver}}{R}\right], &(47e)\end{aligned}$$

where V_{ta} and V_{tr} are along-track and radial speeds of the moving target, respectively.

Finally, (47e) can be further simplified by noting that $V_{hor}\approx V_s$ and $V_{ver}/V_{hor}\ll 1$, yielding

$$\frac{\mathbf{D}^T}{R}(\mathbf{V}_t-\mathbf{V}_s)\approx\frac{D}{R}\left(V_{ta}-V_s\right)\ . \tag{48}$$

Putting everything together, (39f) becomes

$$\begin{aligned}\dot{R}_2=&\dot{R}_1+\frac{D}{V_{hor}}\mathbf{u}^T\mathbf{A}_s+\frac{D}{R}\left(V_{ta}-V_s\right) &(49a)\\=&\dot{R}_1+\frac{D}{R}V_{ta}-\frac{D}{RV_{hor}}\left(V_sV_{hor}-R\mathbf{u}^T\mathbf{A}_s\right) &(49b)\\\approx&\dot{R}_1+\frac{D}{R}V_{ta}-\frac{D}{RV_s}\left(V_s^2-\mathbf{R}^T\mathbf{A}_s\right) &(49c)\\=&\dot{R}_1-\frac{D}{R}\left(\frac{V_e^2}{V_s}-V_{ta}\right) &(49d)\\\approx&\dot{R}_1-\frac{D}{R}\left(V_g-V_{ta}\right)\ , &(49e)\end{aligned}$$

where we make use of $V_e^2\equiv V_s^2-\mathbf{R}^T\mathbf{A}_s$ (13a) and $V_g\approx V_e^2/V_s$. The latter is the velocity of the beam footprint that moves along the surface of the earth and the approximation sign is mainly due to the fact that the satellite orbit is only approximately circular. Therefore, the first-order coefficient of the Taylor expansion of $R_2(t)$ evaluated at time t_0 can be written as

$$\begin{aligned}\dot{R}_2(t_0)=&\dot{R}_1(t_0)-\frac{D}{R(t_0)}\left(V_g-V_{ta}\right) &(50a)\\=&\left(V_{tr}-V_{sr}\right)-\frac{D}{R(t_0)}\left(V_g-V_{ta}\right)\ . &(50b)\end{aligned}$$

Similarly, we derive the second-order coefficient of the Taylor series expansion of $R_2(t)$ by taking the time derivative of (49e), which simply yields $\ddot{R}_2 \approx \ddot{R}_1$. Therefore, the Taylor expansion of $R_2(t)$ (up to the second order) evaluated at arbitrary time t_0 can be written as

$$\begin{aligned} R_2(t) = R(t_0) - D\left[\Phi(t_0) - \Psi\right] + \left[(V_{tr} - V_{sr}) - \frac{D}{R(t_0)}(V_g - V_{ta})\right](t - t_0) \\ + \frac{1}{2}\left(\frac{V_e^2 - 2V_sV_{ta}}{R(t_0)} + A_{tr}\right)(t - t_0)^2 \ . \end{aligned} \tag{51}$$

We are now ready to generalize the moving target range equation (51) for a multi-channel SAR system (i.e. with multiple phase centers):

$$\begin{aligned} R_p(t) = R_0 + (p-1)D(\Psi - \Phi_0) + (V_{tr} - V_{sr})(t - t_0) \\ + \frac{(p-1)D}{R_0}(V_{ta} - V_g)(t - t_0) + \frac{1}{2}\left(\frac{V_e^2 - 2V_sV_{ta}}{R_0} + A_{tr}\right)(t - t_0)^2 \ , \end{aligned} \tag{52}$$

where $R_0 = R(t_0)$, $\Phi_0 = \Phi(t_0)$; V_{tr}, V_{ta}, and A_{tr} are defined for time t_0; $p = 1, 2, 3, \ldots$ (for phase center 1, 2, 3, etc.); V_{sr} depends on V_s and Φ_0 in a predictable way; V_s, V_g, and V_e vary slowly with time and, therefore, may be evaluated anywhere in the neighborhood of t_0.

If we choose t_0 to be the broadside time t_b, then Φ and $V_{sr}(= -V_s\Phi)$ become zero, resulting in

$$\begin{aligned} R_p(t) = R_b + (p-1)D\Psi + V_{tr}(t - t_b) + \frac{(p-1)D}{R_b}(V_{ta} - V_g)(t - t_b) \\ + \frac{1}{2}\left(\frac{V_e^2 - 2V_sV_{ta}}{R_b} + A_{tr}\right)(t - t_b)^2 \ , \end{aligned} \tag{53}$$

where subscript b denotes the broadside time. We can express (52) in a form that is applicable to both spaceborne and airborne geometries as follows:

$$\begin{aligned} R_p(t) =& R_0 + (p-1)D(\Psi - \Phi_0) + (V_{tr} + V_s\Phi_0)T_{pri}(m - m_0) + \ldots \\ & \frac{(p-1)D}{R_0}(V_{ta} - V_g)T_{pri}(m - m_0) + \left(\frac{V_sV_g - 2V_sV_{ta}}{2R_0} + \frac{A_{tr}}{2}\right)T_{pri}^2(m - m_0)^2 \qquad \text{(54a)} \\ =& R_0 + (p-1)D(\Psi - \Phi_0) + V_sT_{pri}\left(\frac{V_{tr}}{V_s} - \Phi_0\right)(m - m_0) + \ldots \\ & \frac{(p-1)DV_sT_{pri}}{R_0}\left(\frac{V_{ta}}{V_s} - \frac{V_g}{V_s}\right)(m - m_0) + \ldots \\ & \frac{V_s^2T_{pri}^2}{2R_0}\left(\frac{V_g}{V_s} - 2\frac{V_{ta}}{V_s} + \frac{R_0}{V_s^2}A_{tr}\right)(m - m_0)^2 \qquad \text{(54b)} \\ =& R_0 + D(\Psi - \Phi_0)(p-1) + D_s(\mu + \Phi_0)(m - m_0) + \ldots \\ & \frac{DD_s}{R_0}(\nu - \beta^2)(p-1)(m - m_0) + \frac{D_s^2}{2R_0}(\beta^2 - 2\nu + \eta)(m - m_0)^2 \ , \qquad \text{(54c)} \end{aligned}$$

where

$$\beta^2 \equiv \frac{V_e^2}{V_s^2} \approx \frac{V_g}{V_s} = \frac{D_g}{D_s} \leq 1 \tag{55a}$$

$$D_g \equiv V_g T_{pri} \tag{55b}$$

$$D_s \equiv V_s T_{pri} \tag{55c}$$

$$\nu \equiv \frac{V_{ta}}{V_s} \tag{55d}$$

$$\mu \equiv \frac{V_{tr}}{V_s} \tag{55e}$$

$$\eta \equiv \frac{R_0}{V_s^2} A_{tr} \tag{55f}$$

$$m_0 \equiv \frac{t_0}{T_{pri}} \ . \tag{55g}$$

T_{pri} is the pulse repetition interval, and m is pulse index in slow time. Notably, the effect of gravitational acceleration is completely absorbed into the parameter $\beta^2 \leq 1$. Clearly, (54c) is also applicable to airborne imaging geometry by setting $\beta^2 = 1$ since, for the airborne case, the platform is assumed to be moving along a straight line and transmitting uniformly spaced pulses. This assumption, however, requires good motion compensation and good control of the pulse repetition frequency (PRF) as a function of ground speed, such that the SAR ground speed V_g is kept the same as its platform speed V_s (or $D_g = D_s$).

Finally, (54c) can be re-expressed for the case when $t_0 = t_b$ as follows:

$$\begin{aligned} R_p(t) = R_b + D\Psi(p-1) + D_s\mu(m-m_b) + \frac{DD_s}{R_b}(\nu-\beta^2)(p-1)(m-m_b) \\ + \frac{D_s^2}{2R_b}(\beta^2 - 2\nu + \eta)(m-m_b)^2 \ . \end{aligned} \tag{56}$$

As stated earlier, Ψ can be compensated by channel balancing and, therefore, may be ignored. Often, η (the normalized target radial acceleration) is also ignored under the assumption that the moving target is not accelerating. In this case, (56) yields

$$R_p(t) = R_b + D_s\mu(m-m_b) + \frac{DD_s}{R_b}(\nu-\beta^2)(p-1)(m-m_b) + \frac{D_s^2}{2R_b}(\beta^2 - 2\nu)(m-m_b)^2 \ , \tag{57}$$

where subscript b, again, denotes that the Taylor series expansion is evaluated at the broadside time t_b.

The accuracy of the equations of motion derived above has been tested and validated using the recently acquired RADARSAT-2 MODEX data [5] [6]. As discussed, these equations are applicable to both airborne and spaceborne imaging geometries and serve as a physical basis for further algorithm development.

4 Conclusions

Both high resolution Synthetic Aperture Radar (SAR) processing and SAR Ground Moving Target Indication (SAR-GMTI) require that a highly accurate imaging geometry model be first established. This can be quite easily accomplished for the case of an airborne platform, which is assumed to be moving along a straight line and transmitting uniformly spaced pulses. For an orbiting platform, the gravitational force plays a key role in defining the satellite trajectory and its ground speed and, thus, must be taken into account. Also, equations of motion of a moving target that accurately model a SAR system equipped with multiple apertures (e.g. RADARSAT-2, TerraSAR-X) are evidently absent in the open literature, partly because there were no multi-aperture spaceborne SARs in the unclassified world prior to late 2007.

This short technical memo derives a set of equations of motion that accurately describes a ground moving target in spaceborne multi-channel SAR imaging geometry. The target range history model is derived in the Earth Centered, Earth-Fixed (ECEF) frame of reference using linearization for small angles as a function of the receive phase center.

The accuracy of the equations of motion derived in this short note have been tested and validated using the recently acquired RADARSAT-2 MODEX data. These equations of motion have been shown to be applicable to both airborne and spaceborne imaging geometries and serve as a physical basis for further algorithm development.

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Annex A: Acceleration in the ECEF Frame

In this annex, we would like to derive and express the acceleration vector $\mathbf{A_s}$ of an orbital satellite in the Earth-Centered-Earth-Fixed reference frame. Let ω_e be the earth's angular rotation speed, ignoring precession and mutation. The transformation matrix $\mathbf{\Gamma}_\omega$ from the Earth-Centered-Inertial (ECI) frame to the ECEF frame is then given by

$$\mathbf{\Gamma}_\omega = \begin{bmatrix} \cos(\omega_e t) & -\sin(\omega_e t) & 0 \\ \sin(\omega_e t) & \cos(\omega_e t) & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{A.1}$$

We begin our derivation with the satellite's position vectors $\bar{\mathbf{R}}_s$ and $\mathbf{R}_s$ in the ECI (denoted by the bar notation) and ECEF frames, respectively. The transformation of the satellite position vector between the two reference frames can then be accomplished through the following two equations:

$$\bar{\mathbf{R}}_s = \mathbf{\Gamma}_\omega \mathbf{R}_s \tag{A.2a}$$

$$\mathbf{R}_s = \mathbf{\Gamma}_\omega^T \bar{\mathbf{R}}_s\,. \tag{A.2b}$$

Differentiating (A.2a) and (A.2b) with respect to time, we obtain

$$\begin{aligned} \bar{\mathbf{V}}_s &= \mathbf{\Gamma}_\omega \mathbf{V}_s + \dot{\mathbf{\Gamma}}_\omega \mathbf{\Gamma}_\omega^T \bar{\mathbf{R}}_s \\ &= \mathbf{\Gamma}_\omega \mathbf{V}_s + \omega_e \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{\mathbf{R}}_s \end{aligned} \tag{A.3a}$$

$$\begin{aligned} \mathbf{V}_s &= \mathbf{\Gamma}_\omega^T \bar{\mathbf{V}}_s + \dot{\mathbf{\Gamma}}_\omega^T \bar{\mathbf{R}}_s \\ &= \mathbf{\Gamma}_\omega^T \bar{\mathbf{V}}_s + \dot{\mathbf{\Gamma}}_\omega^T \mathbf{\Gamma}_\omega \mathbf{R}_s \\ &= \mathbf{\Gamma}_\omega^T \bar{\mathbf{V}}_s + \omega_e \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{R}_s \\ &= \mathbf{\Gamma}_\omega^T \bar{\mathbf{V}}_s + \omega_e \begin{bmatrix} R_{sy} \\ -R_{sx} \\ 0 \end{bmatrix}. \end{aligned} \tag{A.3b}$$

To get the satellite acceleration in the ECEF frame, we differentiate again (A.3a) with respect to time:

$$\begin{aligned}
\mathbf{A}_s &= \mathbf{\Gamma}_\omega^T \bar{\mathbf{A}}_s + 2\dot{\mathbf{\Gamma}}_\omega^T \bar{\mathbf{V}}_s + \ddot{\mathbf{\Gamma}}_\omega^T \bar{\mathbf{R}}_s \\
&= \mathbf{\Gamma}_\omega^T \bar{\mathbf{A}}_s + 2\dot{\mathbf{\Gamma}}_\omega^T \mathbf{\Gamma}_\omega \mathbf{V}_s + 2\dot{\mathbf{\Gamma}}_\omega^T \dot{\mathbf{\Gamma}}_\omega \mathbf{R}_s + \ddot{\mathbf{\Gamma}}_\omega^T \mathbf{\Gamma}_\omega \mathbf{R}_s && \text{(A.4a)} \\
&= \mathbf{\Gamma}_\omega^T \bar{\mathbf{A}}_s + 2\omega_e \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{V}_s + \omega_e^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{R}_s && \text{(A.4b)} \\
&= \mathbf{\Gamma}_\omega^T \bar{\mathbf{A}}_s + 2\omega_e \begin{bmatrix} V_{sy} \\ -V_{sx} \\ 0 \end{bmatrix} + \omega_e^2 \begin{bmatrix} R_{sx} \\ R_{sy} \\ 0 \end{bmatrix}. && \text{(A.4c)}
\end{aligned}$$

$\bar{\mathbf{A}}_s$ is the earth's gravitational field (or acceleration), which can be approximated by

$$\bar{\mathbf{A}}_s \approx -\frac{GM_e}{R_s^3} \bar{\mathbf{R}}_s \,, \tag{A.5}$$

where M_e is the mass of the earth and G is the gravitational constant. This approximate result may be insufficient. There are many available models that include higher spherical harmonics [7].

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The modeling of a moving target for a single channel spaceborne SAR geometry has already been accomplished to a high degree of accuracy by Eldhuset *et al.*, but extending the model to include a SAR system that is equipped with multiple apertures (e.g. RADARSAT-2, TerraSAR-X) still requires further work. The purpose of this short technical memo is to do exactly that – to derive a set of equations of motion of a ground moving target for a multi-channel spaceborne SAR. These equations of motion will be shown to be applicable to both airborne and spaceborne multi-channel SAR systems in stripmap mode.

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RADARSAT-2
MODEX
Moving Target Indication
Equations of Motion
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